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The index of a Ginsparg–Wilson Dirac operator

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Abstract

A novel feature of a Ginsparg–Wilson lattice Dirac operator is discussed. Unlike the Dirac operator for massless fermions in the continuum, this lattice Dirac operator does not possess topological zero modes for any topologically-nontrivial background gauge fields, even though it is exponentially-local, doublers-free, and reproduces correct axial anomaly for topologically-trivial gauge configurations.

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In the continuum, the Dirac operator $\gamma_\mu(\partial_\mu + iA_\mu)$ of massless fermions in a smooth background gauge field with nonzero topological charge Q has zero eigenvalues and the corresponding eigenfunctions are chiral. The Atiyah–Singer index theorem [1,2] asserts that the difference of the number of left-handed and right-handed zero modes is equal to the topological charge of the gauge field configuration:

$$n_- - n_+ = Q. \tag{1}$$

However, if one attempts to use the lattice [3] to regularize the theory nonperturbatively, then *not* every Ginsparg–Wilson lattice Dirac operator [4] might possess topological zero modes [1] with index satisfying (1), even though it is exponentially-local, doublers-free, and reproduces correct axial anomaly for topologically-trivial gauge backgrounds. As a consequence, a topologically-trivial lattice Dirac operator might not realize 't Hooft's solution to the $U(1)$ problem in QCD, nor other quantities pertaining to the nontrivial gauge sectors. Nevertheless, from a theoretical viewpoint, it is interesting to realize that one may have the option to turn off the topological zero modes of a Ginsparg–Wilson lattice Dirac operator, without affecting its correct behaviors (axial anomaly, fermion propagator, etc.) in the topologically-trivial gauge sector. In this Letter, I construct an example of such Ginsparg–Wilson lattice Dirac operators, and argue that it does not possess topological zero modes for any topologically-nontrivial gauge configurations satisfying a very mild condition, Eq. (30).

Consider the lattice Dirac operator

$$D = a^{-1} D_c (\mathbb{1} + D_c)^{-1} \tag{2}$$

with

$$D_c = \sum_\mu \gamma_\mu T_\mu, \quad T_\mu = f t_\mu f, \tag{3}$$

$$f = \left(\frac{1}{\sqrt{t^2 + w^2} + w} \right)^{1/2}, \quad t^2 = -\sum_\mu t_\mu t_\mu. \tag{4}$$

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[1] So far, it has been confirmed that overlap Dirac operator [5,6] and its generalization [7] can possess topological zero modes with index satisfying (1), on a finite lattice.

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Here $\gamma_\mu t_\mu$ is the naive lattice fermion operator and $-w$ is the Wilson term with a negative mass $-1/2$,

$$t_\mu(x,y) = \frac{1}{2}\left[U_\mu(x)\delta_{x+\hat{\mu},y} - U_\mu^\dagger(y)\delta_{x-\hat{\mu},y}\right], \tag{5}$$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu^\dagger & 0 \end{pmatrix}, \tag{6}$$

$$\sigma_\mu\sigma_\nu^\dagger + \sigma_\nu\sigma_\mu^\dagger = 2\delta_{\mu\nu}, \tag{7}$$

$$w(x,y) = \frac{1}{2} - \frac{1}{2}\sum_\mu\left[2\delta_{x,y} - U_\mu(x)\delta_{x+\hat{\mu},y} - U_\mu^\dagger(y)\delta_{x-\hat{\mu},y}\right], \tag{8}$$

where the color and Dirac indices have been suppressed. Note that the D_c defined in Eq. (3) can be regarded as a symmetrized version of that constructed in Ref. [8], for vector gauge theories.

First, we examine D in the free fermion limit. In the momentum space, it can be written as

$$D(p) = D_0(p) + i\sum_\mu \gamma_\mu D_\mu(p), \tag{9}$$

where

$$D_0(p) = \frac{1}{a}\left(\frac{f^4(p)t^2(p)}{1+f^4(p)t^2(p)}\right), \tag{10}$$

$$D_\mu(p) = \frac{1}{a}\sin(p_\mu a)\left(\frac{f^2(p)}{1+f^4(p)t^2(p)}\right), \tag{11}$$

$$t^2(p) = \sum_\mu \sin^2(p_\mu a), \tag{12}$$

$$w(p) = \frac{1}{2} - \sum_\mu\left[1-\cos(p_\mu a)\right], \tag{13}$$

$$f^2(p) = \frac{1}{\sqrt{t^2(p)+w^2(p)}+w(p)}. \tag{14}$$

Now using the relation

$$1+f^4(p)t^2(p) = \frac{2\sqrt{t^2(p)+w^2(p)}}{\sqrt{t^2(p)+w^2(p)}+w(p)}, \tag{15}$$

one can reduce (10) and (11) to

$$D_0(p) = \frac{1}{2a}\left(1 - \frac{w(p)}{\sqrt{t^2(p)+w^2(p)}}\right), \tag{16}$$

$$D_\mu(p) = \frac{1}{2a}\frac{\sin(p_\mu a)}{\sqrt{t^2(p)+w^2(p)}}. \tag{17}$$

Evidently, both $D_0(p)$ and $D_\mu(p)$ are analytic functions for all p in the Brillouin zone. (Note that $\sqrt{t^2(p)+w^2(p)}$ is bounded.) Thus

$$D(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} D(p) \tag{18}$$

is exponentially-local in the position space. The exponential locality of D in the free fermion limit immediately suggests that D is also exponentially-local for sufficiently smooth background gauge fields.

In the limit $a \to 0$, $D(p)$ behaves like

$$D(p) \sim i\sum_\mu \gamma_\mu p_\mu + O\left(ap^2\right). \tag{19}$$

Thus it has *correct continuum behavior*.

The free fermion propagator of (9) is

$$D^{-1}(p) = a - ia\sum_\mu \gamma_\mu\left(\frac{\sin(p_\mu a)}{\sqrt{t^2(p)+w^2(p)}-w(p)}\right), \tag{20}$$

which has a simple pole at $p=0$, and does not have any other poles in the Brillouin zone. Thus it is *doublers-free*.

Furthermore, D is γ_5-hermitian,

$$D^\dagger = \gamma_5 D\gamma_5, \tag{21}$$

and it breaks the chiral symmetry according to the Ginsparg–Wilson relation [4]

$$D\gamma_5 + \gamma_5 D = 2aD\gamma_5 D. \tag{22}$$

Thus D satisfies the necessary requirements for a decent lattice Dirac operator.

The GW relation (22) immediately implies that the fermionic action $\bar{\psi}D\psi$ is invariant under the generalized chiral transformation [9]

$$\psi \to \exp\left[i\theta\gamma_5(\mathbb{1}-aD)\right]\psi, \tag{23}$$

$$\bar{\psi} \to \bar{\psi}\exp\left[i\theta(\mathbb{1}-aD)\gamma_5\right], \tag{24}$$

where θ is a global parameter. Consequently, the axial anomaly, $\mathrm{tr}[a\gamma_5 D(x,x)]$, can be deduced from the change of fermion integration measure under the exact chiral transformation (23), (24) and its sum over all sites is equal to the index of D, which is a well-defined integer [9,10]

$$\mathrm{index}(D) = n_- - n_+ = \sum_x \mathrm{tr}\left[a\gamma_5 D(x,x)\right], \tag{25}$$

where the trace "tr" runs over the Dirac and color space. However, the index relation (25) does *not* necessarily imply that D can possess topological zero modes with the index satisfying the Atiyah–Singer index theorem (1). In fact, the GW Dirac operator (2) always gives

$$n_+ = n_- = \sum_x \mathrm{tr}\big[a\gamma_5 D(x,x)\big] = 0, \tag{26}$$

for any topologically-nontrivial gauge background, even though D is exponentially-local, doublers-free, γ_5-hermitian, and has correct continuum behavior.

The argument is as follows. From (21) and (22), we have

$$D^\dagger + D = 2aD^\dagger D = 2aDD^\dagger. \tag{27}$$

Thus D is normal and γ_5-hermitian. Then the eigenvalues of D are either real or in complex conjugate pairs. Each real eigenmode has a definite chirality, but each complex eigenmode has zero chirality. Further, the sum of the chirality of all real eigenmodes is zero (chirality sum rule) [11,12]. Now the eigenvalues of D (2) fall on a circle in the complex plane, with center $((2a)^{-1}, 0)$ on the real axis, and radius of length $(2a)^{-1}$. Then the chirality sum rule reads

$$n_+ - n_- + N_+ - N_- = 0, \tag{28}$$

where $n_+(n_-)$ denotes the number of zero modes of positive (negative) chirality, and $N_+(N_-)$ the number of nonzero (eigenvalue a^{-1}) real eigenmodes of positive (negative) chirality.

The chirality sum rule (28) asserts that each topological zero mode must be accompanied by a nonzero real eigenmode with opposite chirality, and vice versa. (Note that both topological zero modes and their corresponding nonzero real eigenmodes are *robust* under local fluctuations of the gauge background, thus one can easily distinguish them from those trivial zero and nonzero real eigenmodes which are unstable under local fluctuations of the background.)

It follows that if D cannot have any nonzero real eigenmodes in topologically nontrivial gauge backgrounds, then D cannot possess any topological zero modes.

From (2), any zero mode of D is also a zero mode of D_c, and vice versa. However, a nonzero real (eigenvalue a^{-1}) eigenmode of D corresponds to a pole (singularity) in the spectrum of D_c, since

$$D_c = D(\mathbb{1} - aD)^{-1}, \tag{29}$$

which is the inverse transform of (2).

Therefore, if the spectrum of D_c does *not* contain any poles (singularities) for a topologically-nontrivial gauge background, then D *cannot* have any nonzero real eigenmodes, thus *no* topological zero modes.

Now we consider topologically-nontrivial gauge configurations satisfying the condition

$$\det\big(\sqrt{t^2 + w^2} + w\big) \neq 0. \tag{30}$$

Then f exists, and D_c (3) is well-defined (without any poles). It follows that D (2) cannot have topological zero modes for any topologically-nontrivial gauge configurations satisfying (30).

It should be emphasized that we have *not* found any *robust* nontrivial gauge configuration violating (30), on a finite lattice. Thus, it is likely that the measure of the nontrivial gauge configurations *not* satisfying (30) is *zero*.

From (26), the topological triviality of D (2) implies that it cannot reproduce correct axial anomaly for topologically-nontrivial backgrounds. Nevertheless, since D is exponentially-local, doublers-free and has correct continuum behavior, these conditions are sufficient to ensure that it reproduces continuum axial anomaly for topologically-trivial gauge backgrounds [13]. Further, its exact chiral symmetry guarantees that it is void of $O(a)$ artifacts, and is not plagued by the notorious problems (e.g., additive mass renormalization, mixings between operators in different chiral representations) which occur to the Wilson–Dirac lattice fermion operator. Therefore, it is interesting to investigate to what extent this GW Dirac operator can provide better chiral properties than the Wilson–Dirac operator, especially in lattice QCD. Moreover, it is interesting to compare the physical observables measured by this GW Dirac operator to those by the overlap Dirac operator, to understand what role is played by the topological zero modes.

Finally, it is instructive to unveil the role of the hermitian operator f in D_c (3). In the free fermion limit, $f(p) \simeq 1 + O(a^2p^2)$ for $p \simeq 0$. Thus it retains the physical mode of the naive lattice fermion operator $\gamma_\mu t_\mu$. On the other hand, at each one of the $(2^d - 1)$ corners of the Brillouin zone (BZ), $f(p) \simeq \infty$ such that $f(p)t_\mu(p)f(p) \simeq \infty$. Thus it decouples all doublers of $\gamma_\mu t_\mu$, even at *finite* lattice spacing. However, the singularities of $D_c(p)$ at $(2^d - 1)$ corners of BZ also render it *nonanalytic*. Nevertheless, they do not affect the analyticity of $D(p)$ since they are cancelled (from the numerator and denominator) in the formula (2). Now if $f(p)$ is analytic for all p except at (2^d-1) corners of BZ, then $D(p)$ (Eqs. (10), (11)) is analytic for all p in BZ. Consequently, $D(x)$ (18) is exponentially-local.

In general, the basic requirements for the hermitian operator f in D_c (3) are:

(i) $f(p) \simeq 1 + O(a^2p^2)$ for $p \simeq 0$.
(ii) At each one of the (2^d-1) corners of the Brillouin zone, $f(p) \simeq \infty$ such that $f(p)t_\mu(p)f(p) \simeq \infty$.
(iii) $f(p)$ is analytic for all p except at (2^d-1) corners of the Brillouin zone.

From this viewpoint, we can infer that if f (4) is replaced by its square (or f^α, $\alpha > 1/2$),

$$f = \frac{1}{\sqrt{t^2+w^2}+w},$$

then the resulting $D(p)$ is also analytic, doublers-free, and has correct continuum behavior.

Presumably, one might also construct entirely new examples of f satisfying above basic requirements (i)–(iii).

In passing, we note that the example defined in Eqs. (2)–(4) is only a special case ($c = 1/2$) of the following GW Dirac operator

$$D = a^{-1}D_c(\mathbb{1} + rD_c)^{-1}, \quad r = \frac{1}{2c}, \tag{31}$$

$$\begin{aligned} D_c &= \sum_\mu \gamma_\mu f t_\mu f, \\ f &= \left(\frac{2c}{\sqrt{t^2+w^2}+w}\right)^{1/2}, \quad t^2 = -\sum_\mu t_\mu t_\mu, \\ w(x,y) &= c - \frac{1}{2}\sum_\mu \big(2\delta_{x,y} - U_\mu(x)\delta_{x+\hat\mu,y} \\ &\qquad - U_\mu^\dagger(y)\delta_{x-\hat\mu,y}\big), \quad 0 < c < 2. \end{aligned} \tag{32}$$

In the free fermion limit, (31) gives

$$D(p) = \frac{c}{a}\left(1 - \frac{w(p)}{\sqrt{t^2(p)+w^2(p)}} + i\sum_\mu \gamma_\mu \frac{\sin(p_\mu a)}{\sqrt{t^2(p)+w^2(p)}}\right),$$

which is analytic, doublers-free, and has correct continuum behavior. Further, there are many viable forms of D_c. For example, a variant of D_c is

$$D_c = \sum_\mu \gamma_\mu f t_\mu f^\dagger,$$

$$f = \left[\left(\sqrt{t^2+w^2} - w\right)\frac{2c}{t^2}\right]^{1/2},$$

which agrees with D_c (32) in the free fermion limit. In this case the condition for gauge configurations (30) should be replaced by

$$\det\left(t^2\right) \neq 0.$$

These lattice Dirac operators all satisfy the necessary requirements for a decent lattice Dirac operator, namely, exponential-locality, doublers-free, correct continuum behavior, γ_5-hermiticity and the Ginsparg–Wilson relation. However, they do *not* possess topological zero modes.

For some years, it has been taken for granted that if a Ginsparg–Wilson lattice Dirac operator has correct axial anomaly for the trivial gauge sector, then it must also reproduce continuum axial anomaly for the nontrivial sectors. However, the lattice Dirac operator (2) provides a counterexample, and suggests that this common conception may *not* be justified.

In general, given a topologically-proper lattice Dirac operator, it can be transformed into a topologically-trivial lattice Dirac operator which is identical to the topologically-proper one in the free fermion limit. On the other hand, given a topologically-trivial GW Dirac operator, it remains an interesting question how to transform it into a topologically-proper one.

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